

Fine structure of Hagedorn transitions

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We study non-perturbative aspects of the Hagedorn transition for IIB string theory in an anti-de Sitter spacetime in the limit that the string length goes to infinity. The theory has a holographic dual in terms of free $\mathcal{N} = 4$ super-Yang-Mills theory on a three-dimensional sphere. We define a double scaling limit in which the width of the transition region around the Hagedorn temperature scales with the effective string coupling with a critical exponent. We show that in this limit the transition is smoothed out by quantum effects. In particular, the Hagedorn singularity of perturbative string theory is removed by summing over two different string geometries: one from the thermal AdS background, the other from a noncritical string background. The associated noncritical string has the scaling of the unconventional branch of super-Liouville theory or a branched polymer.

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1. Introduction and summary

One of the deep mysteries of perturbative string theory is the nature of the Hagedorn transition. In all known string theories with spacetime dimensions greater than two, there exists a critical Hagedorn temperature, T_H , above which the thermal partition function for free strings diverges. The divergence can be attributed to the exponential growth of the number of perturbative string states at high energies. It is generally believed that rather than a limiting temperature for string theory, the Hagedorn temperature signals a transition to a new phase [1,2,3], analogous to the deconfinement transition in a nonabelian gauge theory. From the worldsheet point of view, the Hagedorn temperature is associated with the appearance of new relevant operators in the worldsheet conformal field theory. String theory at a finite temperature can be described by strings propagating in a Euclidean target space with time direction X^0 periodically identified with a period given by the inverse temperature. The winding modes in the Euclidean time direction are irrelevant operators when the temperature T is small. The lowest winding modes become marginal at T_H and become relevant above T_H . The emergence of these new relevant operators signals that the worldsheet theory becomes unstable and most likely will flow to a new infrared fixed point.

By studying the effective field theory near the Hagedorn temperature, Atick and Witten [3] argued that the Hagedorn transition is a first order transition that happens below the Hagedorn temperature seen in perturbation theory. In particular, the genus zero contribution to the free energy is nonzero above the transition, i.e. the free energy is of order g_s^{-2} , where g_s is the string coupling constant. This implies that one can no longer ignore the back reaction of the thermal energy density on the background geometry, since no matter how small g_s is, the back reaction is always at least of order one.

One expects that a precise understanding of the Hagedorn transition and physics of the high temperature phase should give us important insights into the fundamental structure of string theory. Given our very limited understanding of these important questions, it is thus of great interest to find a simpler setting that one could study similar questions in full detail.

Recently, it was found that free $\mathcal{N} = 4$ Yang-Mills theory with gauge group $SU(N)$ on a three sphere S^3 also exhibits a Hagedorn type transition in the limit $N \rightarrow \infty$ [4,5,6]².

² See [7,8,9,10,11] for other recent discussions of phase transitions in weakly coupled Yang-Mills theory.

In the large N limit the number of gauge invariant operators grows exponentially with conformal dimension. This leads to a Hagedorn temperature T_H , given by an order one constant times the curvature of S^3 , above which the thermal partition function diverges in the $N = \infty$ limit. At large but finite N , one finds a weakly first order phase transition in the $1/N$ perturbation theory, with the free energy of order $O(1)$ in the low temperature phase, while of $O(N^2)$ in the high temperature phase.

From the AdS/CFT correspondence [12,13,14], the $\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$ and coupling constant g_{YM} on S^3 describes type IIB superstring theory in $AdS_5 \times S_5$. The string scale l_s and ten-dimensional Newton's constant G are given from Yang-Mills parameters as³

$$\frac{R^4}{l_s^4} = g_{YM}^2 N, \quad \frac{G}{R^8} = N^{-2}, \quad G = g_s^2 l_s^8 = l_p^8, \quad g_s = g_{YM}^2$$

where R is the curvature radius of the AdS_5 . The $1/N$ t'Hooft expansion in Yang-Mills theory corresponds to the g_s expansion (quantum gravitational corrections) in AdS, and a small t'Hooft coupling $g_{YM}^2 N$ implies a strongly coupled worldsheet. Free $SU(N)$ Yang-Mills theory with large but finite N appears to describe a string theory in $AdS_5 \times S_5$ in the limit

$$g_s \rightarrow 0, \quad l_s \rightarrow \infty, \quad \frac{g_s^2 l_s^8}{R^8} = \text{finite} \propto \frac{1}{N^2} \quad (1.1)$$

i.e. we are working in the regime

$$l_s = \infty \gg R \gg l_p. \quad (1.2)$$

While very little is known about this theory⁴, if we assume the validity of the AdS/CFT correspondence in this extreme case, we conclude that it should have the following properties:

- The theory has a perturbative stringy expansion in terms of genus expansion with an effective expansion parameter G/R^8 .
- The theory has a Hagedorn divergence in perturbation theory due to exponential growth of perturbative stringy states.
- The free energy is $O(1)$ below T_H , while of order G^{-1} above T_H .

³ We omit order one numerical constants.

⁴ It has been conjectured that this theory might be related to a theory of higher spins [4,15,16].

Even though this AdS string theory⁵ seems rather different from our familiar flat space string theories, the features listed above about the Hagedorn transition are qualitatively similar to those in flat spacetime. Here we have the advantage that the theory has a holographic description in terms of a free Yang-Mills theory and thus its Hagedorn transition can be studied in detail non-perturbatively.

The Hagedorn transition for free Yang-Mills theory is a large N phase transition [4,6]. The number of states at a given energy E can be enumerated by counting gauge invariant operators of conformal dimension E . Since the theory is free, this can be done exactly. At energies⁶ $1 \ll E \ll N^2$, one can treat all operators of given dimensions as independent. One finds that the number of states grows as $e^{\frac{E}{T_H}}$ with T_H given by an order one constant. At energies $E \sim N^2$, the counting becomes more complicated, since finite N effects like trace relations between single and multi-trace operators are important. Treating all the gauge invariant operators as independent dramatically overcounts the number of states, resulting in the Hagedorn divergence. At energies $E \gg N^2$, instead of counting gauge invariant operators, it is simpler to treat the system as N^2 species of gluons and quarks. Then standard results for the ideal gas implies that the number of states should be proportional to $e^{cN^{\frac{1}{2}}E^{\frac{3}{4}}}$, which is much slower than the Hagedorn growth. The Hagedorn transition can thus be understood as the crossover from a region in which the gauge invariant operator description is more appropriate to a region in which the gluon-quark description is more appropriate. On a compact space like S^3 , a sharp transition can only happen in the strict $N = \infty$ limit. At large but finite N , the transition arises as an artifact of the large N expansion⁷. The interpolation between the low and high temperature regimes should be completely smooth at finite N and becomes narrower and sharper as N is increased.

Translating the above physical picture to the corresponding string theory in AdS, we conclude that

- The Hagedorn divergence in the bulk string theory should be an artifact of the string perturbation theory. The transition should be smooth non-perturbatively.
- The appearance of the Hagedorn divergence in perturbation theory can be attributed to a “stringy exclusion principle”, i.e. not all perturbative states are independent of one another at high energies.

⁵ See also [17] for some recent discussions of Hagedorn transition in large AdS.

⁶ We will take the radius of S^3 to be 1.

⁷ More precisely, the partition function is an analytic function of temperature T for all N . The non-analyticity in T arises as a result of expansion in $1/N$.

In this paper we study non-perturbative aspects of this Hagedorn transition using Yang-Mills theory. We would like to understand how non-perturbative stringy effects in AdS smooth out the sharp Hagedorn transition in perturbative string theory. In other words, we are interested in knowing how the Hagedorn divergence in free string theory is resolved at finite string coupling ⁸.

We now summarize the main results of the paper. We will argue that the large- N Hagedorn transition for free $\mathcal{N} = 4$ Super-Yang-Mills theory belongs to the same universality class as that of the double-trace unitary matrix model [6]

$$Z(a) = \int dU \exp [a(T) \text{Tr} U \text{Tr} U^\dagger] \quad (1.3)$$

where U is a unitary matrix and $a(T)$ is a function of temperature. (1.3) has a Hagedorn type transition in the large N limit at $a(T) = 1$. Thus the Hagedorn transition in the AdS string theory can be studied by examining the critical behavior of (1.3) near $a = 1$. We study the matrix model (1.3) to all orders in the $1/N$ expansion and find that there exists a double scaling limit

$$a - 1 \rightarrow 0, \quad N \rightarrow \infty, \quad (a - 1)N^{\frac{4}{3}} = \text{finite} \quad (1.4)$$

in which the transition is smoothed out. Expressed in the language of the corresponding AdS string theory the limit is

$$T - T_H \rightarrow 0, \quad \frac{l_{5p}}{R} \rightarrow 0, \quad (T - T_H) \left(\frac{R}{l_{5p}} \right)^2 = \text{finite}, \quad l_{5p}^3 = \frac{l_p^8}{R^5} \quad (1.5)$$

where l_{5p} is the five-dimensional Planck length in AdS_5 . The physical meaning of (1.4)–(1.5) can be understood as follows. At infinite N , there is a sharp phase transition at $T = T_H$. At large but finite N (or R/l_{5p}), the transition is smoothed out into a finite region around T_H whose width scales inversely with N (or R/l_{5p}) with some critical exponent. The above double scaling limit ensures that we stay within this region and decouples nonessential physics.

In the double scaling limit (1.4)–(1.5), one finds a simple physical picture for the Hagedorn transition of the bulk string theory. This picture is reminiscent of the Hawking-Page transition [18,19] in the strong coupling regime. We find that the Hagedorn divergence

⁸ Naively $g_{YM} = 0$ seems to imply that string coupling is also zero. The effective genus expansion parameter $g_s^2 l_s^8 / R^8$ in (1.1) for the bulk string theory is nonzero at finite N . This is the sense that we talk about the theory at finite string coupling throughout the paper.

is resolved by summing over two different classical *stringy* geometries. Since we are working in the $l_s \rightarrow \infty$ limit (1.2) for the bulk string theory, the concept of geometry in terms of classical gravity is not valid here. Nevertheless, in the large N limit, classical stringy geometry, defined by exact worldsheet conformal field theory, is still a valid concept.

We find that near the Hagedorn transition (both above and below), the full partition function of the theory can be written in a form

$$Z = Z_T + e^{-S_L} Z_L . \quad (1.6)$$

Z_T can be interpreted as the partition function for strings in an AdS background with imaginary time-direction periodically identified (we will call it thermal AdS), which contains a Hagedorn divergence. Z_L is the partition function of a noncritical string theory which emerges only as the Hagedorn temperature is approached. We find that it has the same scaling as the unconventional branch of the $\mathcal{N} = 1$ super-Liouville theory or a branched polymer. S_L can be interpreted as the difference in the classical string field action between the above two string backgrounds. When $T < T_H$, e^{-S_L} is exponentially small in large N and the thermal AdS dominates. At $T \approx T_H$, S_L is of order one and the two string backgrounds are equally important⁹. In particular, the Hagedorn divergence in Z_T is cancelled by a similar divergence in Z_L . In this regime, the theory does not have a (stringy) geometric interpretation. When $T > T_H$, e^{-S_L} becomes exponentially large and the noncritical string background completely dominates. Z_L is tachyonic below T_H while the thermal AdS is tachyonic above T_H .

We also find some interesting non-renormalization properties for Z_T and Z_L . Z_T contains only a genus one contribution with all higher loop contributions vanishing. Below T_H , Z_L also does not receive perturbative corrections beyond one-loop. Above T_H , Z_L does receive perturbative contributions to all loops.

The plan of the paper is as follows. In section 2 we discuss the large N Hagedorn transition of $\mathcal{N} = 4$ SYM theory at infinite N . In section 3 we discuss the Hagedorn transition at finite N in Yang-Mills theory. We find that there exists a double scaling limit in which the partition function smoothly interpolates between the low and high temperature regimes. We discuss the interpretation of Yang-Mills calculation in terms of string theory in AdS. We conclude with discussions of implications of our findings in section 4. In Appendix A, we discuss a generalization of (1.3) which includes both single and double trace terms. This arises when one includes fundamental matter in the theory. We show that the double scaling limit of the model is the same as that of the single-trace unitary matrix model.

⁹ More precisely, we mean $T - T_H \sim O(N^{-2})$.

2. Free large N Yang-Mills theory on S^3

In this section we briefly review the Hagedorn transition for a free $SU(N)$ Yang-Mills theory (with adjoint matter) on S^3 in the infinite N limit [4,6]. While the results presented here are not new, our derivation of the phase transition is new. The basic idea is to integrate out all fields in the theory except for the zero mode of the Polyakov loop. The partition function is then reduced to a unitary matrix integral.

Expanding all fields in Yang-Mills theory in terms of harmonics on S^3 , to the lowest order in g_{YM} , the theory reduces to a quantum mechanics problem of free harmonic oscillators

$$\mathcal{L} = \frac{1}{2} \sum_a \text{Tr} [(D_t M_a)^2 - \omega_a^2 M_a^2] \quad (2.1)$$

where M_a are $N \times N$ Hermitian matrices and the sum is over all field types and their Kaluza-Klein descendants on S^3 . ω_a is the frequency for each mode. We will take the radius of S^3 to be one. The covariant derivative in (2.1) is given by

$$D_t M_a = \partial_t M_a - i[A, M_a]$$

A comes from the zero mode (i.e. the mode independent of coordinates on S^3) of the time component of the gauge field and is not dynamical. Its equation of motion imposes the constraint that physical states must be $SU(N)$ singlets, i.e. Gauss law on S^3 . The partition function of the theory at finite temperature can then be obtained by integrating out all fields in (2.1) except for A . Projecting into the singlet sector of the standard results for harmonic oscillators one finds

$$Z = \int DU \prod_a (\det_{adj} (1 - \epsilon_a e^{-\beta \omega_a} U))^{-\epsilon_a} \quad (2.2)$$

where U is an $SU(N)$ matrix whose integration imposes the singlet condition¹⁰. It can also be interpreted as the Wilson line for A around the Euclidean time circle (Polyakov loop). \det_{adj} denotes the determinant in the adjoint representation and $\epsilon_a = 1$ (-1) for bosonic (fermionic) M_a . Equation (2.2) can be more conveniently written as

$$Z = \mathcal{C}(\beta) \int DU \exp \left(\sum_{n=1}^{\infty} \frac{w_n}{n} \text{Tr} U^n \text{Tr} U^{-n} \right) \quad (2.3)$$

¹⁰ Note that we normalize the measure DU so that $\int DU = 1$.

with

$$w_n = z_B(n\beta) + (-1)^{n+1} z_F(n\beta) . \quad (2.4)$$

$z_B(\beta)$, $z_F(\beta)$ are the single particle partition function for the bosonic and fermionic sector respectively, e.g.

$$z_B(\beta) = \sum_{a, \text{bosons}} e^{-\beta\omega_a}, \quad z_F(\beta) = \sum_{a, \text{fermions}} e^{-\beta\omega_a} . \quad (2.5)$$

The prefactor $C(\beta)$ in (2.3) is given by

$$\log \mathcal{C}(\beta) = \beta E_0 - \sum_{n=1}^{\infty} \frac{w_n}{n} . \quad (2.6)$$

E_0 is the zero-point energy of the theory on S^3 and the second term arises because the group is $SU(N)$ rather than $U(N)$. For free $\mathcal{N} = 4$ SYM theory on S^3 , we have [4,6]

$$z_B = \frac{6x + 12x^2 - 2x^3}{(1-x)^3}, \quad z_F = \frac{16x^{\frac{3}{2}}}{(1-x)^3}, \quad x = e^{-\beta} . \quad (2.7)$$

The large N limit of the matrix integral (2.3) can be evaluated using a saddle point approximation [20,21]. Depending on the values of w_n , it could have a complicated phase structure [22]. For $\mathcal{N} = 4$ SYM, one finds from (2.7) that parametrically

$$\frac{w_n}{w_1} \ll 1, \quad n > 1 \quad (2.8)$$

and only two phases arise [4,6] with the critical temperature T_H given by the solution of the equation $w_1(T_H) = 1$. (2.8) implies that one can approximate (2.3) by the first term in the exponential

$$Z \approx \int DU e^{w_1 \text{Tr} U \text{Tr} U^\dagger} \quad (2.9)$$

In particular, since the critical behavior of (2.3) near $w_1 = 1$ is controlled by the first term, one expects that the two models (2.3) and (2.9) should have the same critical behavior. Since we are interested in the transition region, for the rest of the paper we will focus on (2.9).

The matrix model (2.9) can be solved to all orders by introducing a Lagrange multiplier to eliminate the double trace term in the exponential¹¹

$$\begin{aligned} Z(a) &= \int DU \exp(a \text{Tr} U \text{Tr} U^\dagger) \\ &= \frac{1}{2\pi a} \int DU d\lambda d\bar{\lambda} \exp \left[-\frac{1}{a} \lambda \bar{\lambda} + \lambda \text{Tr} U + \bar{\lambda} \text{Tr} U^\dagger \right] . \end{aligned} \quad (2.10)$$

¹¹ I thank M. Douglas and V. Kazakov for discussions on this point. A similar method was also used in the context of double trace deformation of hermitian matrix models by Klebanov and collaborators [23,24].

Absorbing¹² the phase of λ into U and letting

$$|\lambda| = \frac{1}{2}Ng$$

we find that

$$Z(a) = \frac{N^2}{2a} \int_0^\infty g dg \exp \left(-\frac{N^2 g^2}{4a} + N^2 F(g) \right) \quad (2.11)$$

where $F(g)$ is given by the unitary matrix integral

$$e^{N^2 F(g)} = \int DU \exp \left(\frac{1}{2} Ng (\text{Tr} U + \text{Tr} U^\dagger) \right). \quad (2.12)$$

Now $Z(a)$ reduces to an integral of the partition function of a unitary matrix model weighted by a Gaussian factor over its coupling constant.

The large N expansion of matrix integral (2.12) is well known [21,25,26] and the leading order term is given by

$$F(g) = \begin{cases} \frac{g^2}{4} & g \leq 1 \\ g - \frac{1}{2} \log g - \frac{3}{4} & g > 1 \end{cases} \quad (2.13)$$

The discontinuity in the third derivative of $F(g)$ at $g = 1$ is the Gross-Witten large N phase transition [21,27]. The $g < 1$ phase corresponds to a saddle point of (2.12) at which the eigenvalues $e^{i\theta_i}, i = 1, \dots, N$ of U are distributed around the whole unit circle. In particular at $g = 0$, the distribution is uniform. In the $g > 1$ phase, the eigenvalue distribution develops a gap and does not cover the whole unit circle. When $g \rightarrow \infty$ all eigenvalues are localized at $\theta_i = 0$.

The leading order result for $Z(a)$ is now simply given by the saddle point of the g -integral (2.11). The Gaussian weight factor picks the eigenvalue distributions for different values of a from those of (2.12). We introduce

$$Q(a, g) = -\frac{g^2}{4a} + F(g) \quad (2.14)$$

and look for local maximums of Q . We find that:

1. For $a < 1$, $Q(a, g)$ is a monotonically decreasing function of g with maximum at

$$g = 0, \quad Q(a, 0) = 0 \quad (2.15)$$

¹² This turns the $SU(N)$ integral into an $U(N)$ integral.

Expanding $Q(a, g)$ around $g = 0$ and performing the Gaussian integral we find that

$$\log Z(a) = -\log(1 - a) + \dots \quad (2.16)$$

2. For $a > 1$, $Q(a, 0) = 0$ is now a local minimum. The function has a maximum at

$$g_0 = \frac{1}{1 - w} > 1, \quad w = \sqrt{1 - \frac{1}{a}} \quad (2.17)$$

with

$$\log Z(a) = N^2 Q(g_0) = \frac{N^2}{2} \left(\frac{w}{1 - w} + \log(1 - w) \right) + \dots \quad (2.18)$$

For $\mathcal{N} = 4$ SYM, $a(T)$ is a monotonically increasing function of T

$$a(T) = w_1 = \frac{2x(3 - \sqrt{x})}{(1 - \sqrt{x})^3}, \quad x = e^{-\beta} \quad (2.19)$$

with the critical temperature at

$$w_1(\beta_H) = 1, \quad \beta_H = 2 \log(2 + \sqrt{3}) . \quad (2.20)$$

There is no contribution of $O(N^2)$ in the low temperature phase partition function (2.16) and the genus one term diverges as the critical temperature is approached,

$$\log Z \approx -\log(\beta - \beta_H) + \text{const}, \quad T \rightarrow T_H \quad (2.21)$$

It follows that the density of states around T_H is (which can be found by a Laplace transform of (2.16))

$$\Omega(E) \approx \text{const } e^{\beta_H E} (1 + O(1/E^2)) \quad (2.22)$$

Note that (2.21) and (2.22) are typical of Hagedorn behavior. In fact they have exactly the same form as those in flat space string theory with all spatial directions compactified (see e.g. [28,29]). At order $O(N^2)$ there is a weakly first order transition at $a = 1$. The high temperature phase has nonzero genus zero contribution¹³.

In next section we will look at finite N corrections to the above results. We would like to understand how finite N corrections resolve the singularity in (2.21) and smooth

¹³ Note that although (2.18) is continuous as $a \rightarrow 1$, the second derivative of $\log Z(a)$ (i.e. specific heat) becomes singular.

out the transition. We conclude this section by making a few remarks on the physical interpretation of U in (2.3).

1. U can be understood as the temporal Wilson line for A around the time circle τ (Polyakov loop)

$$U^n = P \exp \left(i \int_0^{n\beta} A d\tau \right) \quad (2.23)$$

The theory has a Z_N symmetry,

$$U \rightarrow e^{\frac{2\pi i k}{N}} U, \quad k \in \mathbf{Z} \quad (2.24)$$

coming from gauge transformations which are periodic up to an element of the center Z_N . In terms of the eigenvalues $e^{i\theta_i}, i = 1, \dots, N$ of U the Z_N symmetry corresponds to discrete translations $\theta_i \rightarrow \theta_i + \frac{2\pi k}{N}$. In fact (2.3) has a larger “accidental $U(1)$ symmetry”, since it was obtained from a free theory. The correlation functions of the theory, however, do not have to respect this larger symmetry, since the measure DU for $SU(N)$ is only Z_N invariant.

2. In the large N limit, it is convenient to introduce density of eigenvalues

$$\rho(\theta) = \frac{1}{N} \sum_{i=1}^N \delta(\theta - \theta_i), \quad -\pi \leq \theta < \pi \quad (2.25)$$

Then

$$\frac{1}{N} \text{Tr} U^n = \rho_n = \int_{-\pi}^{\pi} d\theta \rho(\theta) e^{in\theta}$$

and (2.3) can be written as

$$Z = \mathcal{C} \int [D\rho] e^{-V[\rho]} \quad (2.26)$$

where $V(\rho)$ has the form

$$V[\rho] = N^2 \sum_{n=1}^{\infty} \frac{1 - w_n}{n} |\rho_n|^2 \quad (2.27)$$

ρ_n can be interpreted in the dual AdS string theory as the n -th winding modes around the time circle. From (2.27) and (2.20), ρ_1 become massless at the Hagedorn temperature and tachyonic above T_H . The Hagedorn divergence (2.21) is precisely due to that ρ_1 becoming massless [4,6]. In the low temperature phase, the eigenvalue distribution is given by that of (2.12) at $g = 0$, which is

$$\rho(\theta) = \frac{1}{2\pi} .$$

In the high temperature phase, $\rho(\theta)$ develops a gap.

3. The Z_N symmetry (2.24) becomes an $U(1)$ in the large N limit, corresponding to

$$\theta \rightarrow \theta + \alpha \quad (2.28)$$

Under the transformation (2.28), ρ_n transform as

$$\rho_n \rightarrow \rho_n e^{-in\alpha}$$

It is tempting to interpret the θ -circle as the dual time circle in AdS string theory. However, here $\alpha' = \infty$, it is not clear how to define T-duality. The $U(1)$ symmetry of the original time circle $\tau \rightarrow \tau + \epsilon$ is not manifest in the matrix model. Due to the symmetry (2.28), given a saddle point solution $\rho(\theta)$ of (2.26), $\rho(\theta + \alpha)$ is a solution for any α . Thus there are a continuous family of saddle points in the high temperature phase.

3. Hagedorn transition at finite N

In this section we would like to understand how finite N corrections in Yang-Mills theory resolve the singular behavior in the large N expansion and smooth out the transition. From the AdS/CFT correspondence, this tells us how non-perturbative stringy effects smooth out the singular Hagedorn behavior in perturbation theory for the bulk string theory.

For this purpose, we need to work out the sub-leading corrections to the matrix integral (2.10) near the transition region (i.e. $a \approx 1$). This can be obtained by expanding around the saddle points of

$$\begin{aligned} Z(a) &= \frac{N^2}{2a} \int_0^\infty g dg \exp \left(-\frac{N^2 g^2}{4a} + N^2 F(g) \right) \\ &= \frac{N^2}{2a} \int_0^\infty g dg e^{N^2 Q} \end{aligned} \quad (3.1)$$

using the large N expansion of the unitary matrix model

$$e^{N^2 F(g)} = \int DU \exp \left(\frac{1}{2} N g (\text{Tr} U + \text{Tr} U^\dagger) \right). \quad (3.2)$$

We will first review the results for (3.2).

3.1. Unitary matrix model

The asymptotic expansion of (3.2) in large N is well known [25,26]. Depending on the value of g , the expansion can be divided into three different regions

$$N^2 F(g) \approx \begin{cases} \frac{N^2 g^2}{4} + \frac{1}{2\pi} e^{-2Nf(g)} \sum_{n=1}^{\infty} \frac{1}{N^n} F_n^{(1)} & g < 1 \\ \frac{N^2 g^2}{4} + \sum_{n=0}^{\infty} N^{-\frac{2}{3}n} F_n^{(2)} & g - 1 \sim O(N^{-\frac{2}{3}}) \\ N^2 \left(g - \frac{1}{2} \log g - \frac{3}{4} \right) + \sum_{n=0}^{\infty} N^{-2n} F_{2n}^{(3)} & g > 1 \end{cases} \quad (3.3)$$

where

$$f(g) = \log \left(\frac{1}{g} + \sqrt{\frac{1}{g^2} - 1} \right) - g \sqrt{\frac{1}{g^2} - 1} \quad (3.4)$$

Explicit expressions for various series $F_n^{(1)}, F_n^{(2)}, F_n^{(3)}$ are not important for our discussion below and some important features of (3.3) are:

1. There are no perturbative corrections to $F(g)$ for $g < 1$ beyond the leading term. All corrections are non-perturbative in N .
2. Sub-leading terms in the expansions for $g < 1$ and $g > 1$ become singular as $g \rightarrow 1$,

$$F_n^{(1)} \sim \frac{1}{(1-g)^{\frac{3n}{2}}}, \quad F_n^{(3)} \sim \frac{1}{(g-1)^{3n}}. \quad (3.5)$$

The asymptotic expansions in $1/N$ break down and are replaced by that in intermediate region $|g - 1| \sim N^{-\frac{2}{3}}$.

3. One can define a double scaling limit

$$g = 1 - N^{-\frac{2}{3}} t, \quad N \rightarrow \infty \quad (3.6)$$

which extracts the $F_0^{(2)}$ term from the expansion. $F_0^{(2)}$ satisfies the equation

$$\frac{d^2}{dt^2} F_0^{(2)}(t) = -f^2(t) \quad (3.7)$$

where $f(t)$ in turn satisfies the Painleve II equation

$$\frac{1}{2} f''(t) = f^3 + t f(t) \quad (3.8)$$

Higher order terms $F_n^{(2)}$ can be shown to satisfy more complicated differential equations. When $|t| \gg 1$, $F_0^{(2)}$ has the following asymptotic expansion

$$F_0^{(2)} = \begin{cases} \frac{t^3}{6} - \frac{1}{8} \log(-t) - \frac{3}{128t^3} + \frac{63}{1024t^6} + \dots & -t \gg 1 \\ \frac{1}{2\pi} e^{-\frac{4\sqrt{2}}{3} t^{\frac{3}{2}}} \left(-\frac{1}{8\sqrt{2}t^{\frac{3}{2}}} + \frac{35}{384t^3} - \frac{3745}{18432\sqrt{2}t^{\frac{9}{2}}} + \dots \right) & t \gg 1 \end{cases} \quad (3.9)$$

Note that $F_0^{(2)}$ is a smooth function of t and interpolates smoothly between $g > 1$ ($t < 0$) phase and $g < 1$ ($t > 0$) phase.

4. It was argued in [30] that $F_0^{(2)}(t)$ describes the full partition function of the type 0B theory in $d = 0$ dimension, i.e. pure 2-d supergravity. The parameter t is proportional to the cosmological constant μ in the super-Liouville interaction.

3.2. Critical behavior around $a \approx 1$ and a double scaling limit

We are interested in understanding the critical behavior of the matrix model (3.1) near the transition point $a \approx 1$. We emphasize that the integral (3.1) is manifestly finite for all values of a . The Hagedorn divergence arises as a result of doing the large N expansion. To see how the Hagedorn divergence is cancelled in the finite N theory, it is convenient to split the integral in (3.1) into

$$\begin{aligned} Z &= \int_0^1 dg g e^{-\frac{N^2 g^2}{4}(\frac{1}{a}-1)} + Z_1 + Z_2 \\ &= \frac{1}{1-a} \left(1 - e^{-\frac{N^2}{4a}(1-a)}\right) + Z_1 + Z_2 \end{aligned} \quad (3.10)$$

with

$$\begin{aligned} Z_1 &= \frac{N^2}{2a} \int_0^1 g dg e^{-\frac{N^2 g^2(1-a)}{4a}} \left(e^{\tilde{F}} - 1\right) \\ Z_2 &= \frac{N^2}{2a} \int_1^\infty g dg e^{N^2 Q} \end{aligned} \quad (3.11)$$

where $\tilde{F}(g)$ is defined for $g < 1$

$$\tilde{F} = N^2 F(g) - \frac{g^2 N^2}{4} \quad (3.12)$$

In (3.10) we separated the integration for g into $g < 1$ and $g > 1$ regions motivated from (3.3). We further isolated from the $g < 1$ part of the integral the term $\frac{1}{1-a}$, which gives rise to the Hagedorn divergence (2.16), and a non-perturbative term

$$-\frac{1}{1-a} e^{-N^2 f(a)}, \quad f(a) = \frac{1-a}{4a}$$

which precisely cancels the divergence at $a = 1$. This latter term is not seen in perturbation theory. One can also readily check that both Z_1 and Z_2 are convergent integrals for all a . They can be evaluated by saddle point approximations. Notice that \tilde{F} defined in (3.12) is exponentially small except near $g = 1$ (see (3.3) and (3.4)). When $a < 1$, one finds that Z_2

is exponentially smaller than Z_1 and Z_1 is dominated by a saddle point which approaches 1 as $a \rightarrow 1_-$. When $a > 1$, Z_1 is exponentially smaller than Z_2 and Z_2 is dominated by the saddle point (2.17) which again approaches 1 as $a \rightarrow 1_+$. It thus follows that the critical behaviors of the theory around $a = 1$ can be obtained by evaluating (3.11) around the region $g \approx 1$.

To find the scaling region, let us first look at the limit of $a \rightarrow 1$ from the high temperature side (2.18). As $a \rightarrow 1_+$, we find that

$$\begin{aligned} g_0 &= 1 + \sqrt{a-1} + O(a-1), \\ \log Z(a) &= N^2 \left(\frac{a-1}{4} + \frac{1}{3}(a-1)^{\frac{3}{2}} + O((a-1)^2) \right) \end{aligned} \quad (3.13)$$

Although $\log Z(a)$ is continuous as $a \rightarrow 1_+$, its second derivative with respect to a (i.e. specific heat) becomes singular in the limit. The leading non-analytic term $N^2(a-1)^{\frac{3}{2}}$ in $\log Z(a)$ suggests a scaling $a-1 \sim O(N^{-\frac{4}{3}})$. This also follows from $a-1 \approx (g_0-1)^2$ and the scaling of the unitary model (3.6). This motivates us to define a double scaling limit

$$1-a = N^{-\frac{4}{3}}s, \quad N \rightarrow \infty. \quad (3.14)$$

With a change of variable

$$g = 1 - N^{-\frac{2}{3}}t, \quad (3.15)$$

the exponent in (3.1) can be written as

$$\begin{aligned} N^2 Q &= \frac{N^2 g^2}{4} \left(1 - \frac{1}{a} \right) + F_0^{(2)} + N^{-\frac{2}{3}} F_1^{(2)} + \dots \\ &= -\frac{1}{4} N^{\frac{2}{3}} s + \frac{1}{2} s t + F_0^{(2)}(t) + O(N^{-\frac{2}{3}}) \end{aligned} \quad (3.16)$$

After some simple algebra, one finds that

$$\begin{aligned} Z &= \frac{N^{\frac{4}{3}}}{s} \left(1 - e^{-\frac{1}{4} N^{\frac{2}{3}} s} \right) + N^{\frac{4}{3}} e^{-\frac{1}{4} N^{\frac{2}{3}} s} \sum_{n=0}^{\infty} N^{-\frac{2n}{3}} B_n(s) \\ &= \frac{N^{\frac{4}{3}}}{s} \left(1 - e^{-\frac{1}{4} N^{\frac{2}{3}} s} \right) + N^{\frac{4}{3}} e^{-\frac{1}{4} N^{\frac{2}{3}} s} B_0(s) + \dots \end{aligned} \quad (3.17)$$

where B_n can be obtained from $F_n^{(2)}$. The leading term $B_0(s)$ is given by

$$B_0(s) = \frac{1}{2} \int_{-\infty}^{\infty} dt e^{\frac{1}{2} s t} \left(e^{F_0^{(2)}} - \theta(t) \right) \quad (3.18)$$

where $\theta(t)$ is the step function. From (3.18) and (3.9) $B_0(s)$ is a completely smooth function of s including $s = 0$. Higher order terms $B_n, n > 0$ are increasingly complicated and we have not looked at them in detail. Our procedure should guarantee that all of them are well defined and smooth functions of s . They are not relevant in the double scaling limit (3.14).

In the double scaling limit (3.14), keeping only the leading term in (3.17), we have

$$Z = \frac{N^{\frac{4}{3}}}{s} + N^{\frac{4}{3}} e^{-\frac{1}{4} N^{\frac{2}{3}} s} \tilde{B}_0(s) \quad (3.19)$$

where we have introduced

$$\tilde{B}_0(s) = B_0(s) - \frac{1}{s}. \quad (3.20)$$

Note that as $s \rightarrow 0$, \tilde{B}_0 is singular

$$\tilde{B}_0(s) \sim -\frac{1}{s} + \dots \quad (3.21)$$

This singularity precisely cancels with that of the first term in (3.19). For $s < 0$, $\tilde{B}(s)$ can also be written as

$$\tilde{B}_0(s) = \frac{1}{2} \int_{-\infty}^{\infty} dt e^{\frac{1}{2} st + F_0^{(2)}(t)} \quad (3.22)$$

while for $s > 0$

$$\tilde{B}_0(s) = \frac{1}{2} \int_{-\infty}^{\infty} dt e^{\frac{1}{2} st} \left(e^{F_0^{(2)}} - 1 \right) \quad (3.23)$$

One can easily generalize the above discussions to unitary models with both single trace and double trace terms, e.g.

$$Z(a, \mu) = \int dU \exp \left[a \text{Tr} U \text{Tr} U^\dagger + \frac{1}{2} N (\mu \text{Tr} U + \bar{\mu} \text{Tr} U^\dagger) \right] \quad (3.24)$$

where U is a unitary matrix. (3.24) is relevant when we include N_f matter fields in the fundamental representation of the gauge group with N_f/N to be finite¹⁴. One can show that the critical behaviors of (3.24) are in fact the same as those of the single-trace unitary matrix models. This implies there is no Hagedorn type transition for (3.24) [9]. Some details can be found in Appendix A.

¹⁴ The actual matrix model resulting by integrating out all fields except for the zero modes of the Wilson loop is more complicated. But again one can argue that (3.24) is enough for understanding the critical behavior of the theory.

3.3. AdS interpretation

The partition function Z (3.19) has the following form

$$Z = Z_T + e^{-S_L} Z_L \quad (3.25)$$

with

$$\begin{aligned} \log Z_T &= -\log s + \text{const} \\ \log Z_L &= \log \tilde{B}_0(s) + \text{const} \end{aligned} \quad (3.26)$$

and

$$S_L = N^{\frac{2}{3}} s = N^2(1 - a(T)) \approx N^2 a'(T_H)(T - T_H) \quad (3.27)$$

Z_T is due to the saddle at $g = 0$ (uniform distribution of matrix eigenvalues) and Z_L comes from $Z_1 + Z_2$ at $g \approx 1$.

We would like to interpret equation (3.25) in terms of contributions from two different *classical stringy geometries* in the corresponding AdS string theory. The free Yang-Mills theory we are working with corresponds to the $l_s \rightarrow \infty$ limit (1.1) for the bulk string theory. Thus the concept of geometry in terms of classical gravity is not valid here. Nevertheless, since we are working in the large N limit, classical stringy geometry, as defined by conformally invariant sigma-model on the worldsheet, is still a valid concept. In particular, saddle points (local maxima) in the matrix model (2.3) should correspond to worldsheet conformal field theories. The saddle points of (2.3) in turn coincide with those of (3.1) in the scaling region (3.14) we are interested in here. Thus we shall interpret (3.25) as summing over two classical string backgrounds. S_L can be interpreted as the difference in the classical string field action for two backgrounds and Z_T, Z_L are partition functions around each background. Clearly $\log Z_T$ should describe string theory in the thermal AdS geometry, i.e. AdS with the time direction periodically identified. The precise string theory interpretation of Z_L is not clear to us. One possible identification is that $\log Z_L$ describes the *unconventional branch* of the super-Liouville theory with s identified as the cosmological constant. In the next subsection we will also mention other possibilities.

The physical picture reflected from (3.25) can be summarized as follows. We first divide our discussions into three regions and then comment on some general aspects.

1. $T < T_H$, $s > 0$ but not too close to zero. In this region the Z_T (thermal AdS) dominates and Z_L is exponentially suppressed. Note that Z_T does not contain perturbative corrections beyond one loop. Strictly speaking, Z_L does not correspond to a local maximum of the full matrix model (3.1). Nevertheless, we find it natural to

interpret it as corresponding to a conformal field theory on the string theory side. This theory appears to be tachyonic due to the negative sign in equation (3.21). For $s \gg 1$ we can evaluate (3.23) as an asymptotic series in $1/s$. Surprisingly, we find that the asymptotic series terminate beyond the Gaussian integration

$$\log \tilde{B}_0 = \frac{s^3}{192} - \frac{5}{2} \log s + C' + i\pi + \left(\text{terms nonperturbative in } \frac{1}{s} \right) \quad (3.28)$$

where C' is a real constant. This implies that $\log Z_L$ does not receive perturbative corrections beyond one-loop. The imaginary term $i\pi$ in (3.28) again indicates that the theory contains tachyonic modes. More precisely, it has the behavior of a complex tachyon in 0-Euclidean dimension.

2. The transition region: $s \sim O(N^{-\frac{2}{3}})$, i.e. $T - T_H \sim O(N^{-2})$. In this region, the two terms in (3.25) are of comparable strength. Since two backgrounds contribute equally in this regime, the full string theory does not have a well defined (stringy) geometric interpretation. The $1/s$ Hagedorn divergence of Z_T is cancelled by a similar divergent term in Z_L (see (3.21)). We remarked below (2.27) that the Hagedorn divergence in Z_T can be understood as ρ_1 becoming massless. What is the interpretation of the $1/s$ singularity for Z_L ? There are two possible interpretations. One is that (3.21) arises from the volume factor of the Liouville theory, i.e. the theory becomes effectively noncompact when $s \rightarrow 0$ ¹⁵. A second interpretation is that Z_L develops a 0-dimensional complex massless mode (let us call it σ_1) as $s \rightarrow 0$, given that the theory is tachyonic for $s > 0$ and not tachyonic for $s < 0$. If true, it means that for $s > 0$, σ_1 is tachyonic, while for $s < 0$, ρ_1 becomes tachyonic.
3. $T > T_H$, $s < 0$ but not too close to zero. The saddle at $g \approx 1$ becomes the local maximum of the full integral. Z_L is exponentially large and dominates. $g = 0$ is no longer a local maximum and Z_T is not physically meaningful¹⁶. When $-s \gg 1$, (3.22) can be evaluated as asymptotic expansions in $1/s$ using (3.9)

$$\log \tilde{B}_0 = \frac{(-s)^{\frac{3}{2}}}{3} + C - \frac{5}{16} \log(-s) + \frac{35}{96(-s)^{\frac{3}{2}}} + \frac{245}{256(-s)^6} + O\left(\frac{1}{(-s)^{\frac{9}{2}}}\right). \quad (3.29)$$

Treating $(-s)^{-\frac{3}{4}}$ as the effective string coupling constant, $\log Z_L$ contains perturbative corrections to all orders.

¹⁵ Of course, the numerical constant multiplying the volume factor has to match (3.21).

¹⁶ Since we have omitted terms in (3.17) much bigger than Z_T .

4. S_L (3.27), which we computed explicitly using the dual Yang-Mills theory, can be interpreted to be the difference of the classical string field action of two backgrounds. S_L cannot be computed in a first quantized formalism with strings moving in a fixed background. In a second quantized formalism, like string field theory, it is natural to expect S_L to be computable.
5. Our discussion above was restricted to the region $T - T_H \sim O(N^{-\frac{4}{3}})$. Going beyond this region requires understanding the full matrix model (2.3), which is a rather non-trivial task. We have studied the truncated model (2.10) to all orders in N in detail for general a . We find that for $T < T_H$, there are no perturbative corrections to the partition function of the thermal AdS background beyond one-loop. We expect this to be true for the full matrix model (2.3), given the quadratic nature of the action. We also find that in (2.10) there are non-perturbative corrections of the form

$$e^{-N \log N} \tag{3.30}$$

and

$$e^{-N^2 f(T)} \tag{3.31}$$

(3.31) again suggests contribution of another geometry. However, away from the scaling region, (3.31) is subdominant compared to (3.30), so it is not clear whether it has an unambiguous meaning. While it is tempting to interpret (3.30) as due to D-instantons, it is not clear to us how to understand the $\log N$ factor.

3.4. The unconventional branch of super-Liouville theory

It is a natural question to ask whether $\log \tilde{B}_0$ has an interpretation in terms of a non-critical string theory¹⁷. The scaling behavior of (3.29) corresponds to a string susceptibility exponent given by

$$\gamma_{st} = \frac{1}{2} \tag{3.32}$$

Here we discuss two possible, perhaps related, interpretations.

A well known phase of random matrix models which has the value of exponent (3.32) is a branched polymer phase (see e.g. [31]). The branched polymer interpretation seems to fit with the picture that in the high temperature (deconfinement) phase, quarks and gluons are “liberated”, and the continuous Riemann surface description breaks down [32].

¹⁷ This subsection grew out of discussions with I. Klebanov and J. Maldacena.

The scaling behavior (3.32) also appeared in the double trace deformations of Hermitian matrix models [33,34,35]. Based on Feynman diagrams generated by the double trace term, it was argued in [33] that the matrix model describes a branched polymer phase, when the coefficient of the double trace term is sufficiently large. In our case, the story is less clear since we do not have a good geometric picture of how discretized worldsheets arise in a unitary matrix model. It is also not clear to us how to interpret the scaling behavior of (3.28) for $s > 0$ from this point of view.

We will now offer an alternative interpretation of (3.29), again drawing inspiration from the analogous questions in the context of double trace deformations of Hermitian matrix models. In [33] it was found that a new phase appears when the coefficient of the double trace term takes a particular value. Klebanov and Hashimoto [23,24] gave an interesting interpretation of this new phase. They observed that the scaling behaviors of the new critical points in the presence of double trace deformations can be explained by changing the branch of the Liouville dressing. Our discussion of the double scaling limit, and in particular, the expressions (3.22) and (3.23) are rather similar to those obtained there. Given that $F_0^{(2)}$ in (3.22) and (3.23) is identified with the conventional branch of the $\mathcal{N} = 1$ super-Liouville theory [30], it seems natural to identify $\log \tilde{B}_0$ with the other branch of the super-Liouville theory. We will now check that the scaling in (3.29) is indeed consistent with this proposal with $(-s)$ identified with the Liouville cosmological constant. In Appendix A we study a double trace deformation of the simplest single trace unitary model and show that its critical behavior is the same as the single trace case except for the case we discussed in previous subsections.

The super-Liouville action can be written as (we take $\alpha' = 2$)

$$\begin{aligned} S &= \frac{1}{2\pi} \int d^2\sigma d^2\theta \left[D\Phi \bar{D}\Phi + 2i\mu_0 e^{b\Phi} \right] \\ &= \frac{1}{2\pi} \int d^2\sigma \left[\partial\phi \bar{\partial}\phi + \frac{1}{4}QR\phi + \psi \bar{\partial}\psi + \bar{\psi} \partial\bar{\psi} + \mu^2 b^2 e^{2b\phi} + 2i\mu_0 b^2 \psi \bar{\psi} e^{b\phi} \right] \end{aligned} \quad (3.33)$$

with the central charge given by

$$\hat{c}_L = 1 + 2Q^2$$

and b satisfying

$$Q = b + \frac{1}{b} \quad (3.34)$$

For pure supergravity we need $\hat{c}_L = 10$ which leads to

$$Q = \frac{3}{\sqrt{2}} \quad (3.35)$$

There are two solutions to (3.34),

$$b_- = \frac{1}{\sqrt{2}}, \quad b_+ = \sqrt{2} \quad (3.36)$$

where b_- is the standard branch which satisfies the Seiberg bound. b_+ branch does not correspond to a local operator.

The dependence of (3.33) on μ_0 can be obtained by taking

$$\Phi \rightarrow \Phi - \frac{1}{b} \ln \mu_0$$

and we find the free energy on a genus h surface is

$$F_h \sim \mu_0^{\chi \frac{Q}{2b}}$$

For the branch b_- we have

$$F_h \sim \mu_0^{\frac{3}{2}(2-2h)} \quad (3.37)$$

The scaling agrees with that of the first line of (3.9) if we identify $\mu_0 > 0$ with $-t$. For $\mu_0 < 0$, F_h are identically zero since there are only non-perturbative corrections in the second line of (3.9). For the other branch b_+ we have

$$F_h \sim \mu_0^{\frac{3}{4}(2-2h)} . \quad (3.38)$$

This agrees with (3.29) if we identify $-s \propto \mu_0 > 0$. It seems natural to identify (3.28) with the same theory with $\mu_0 < 0$. However, the s^3 term in (3.28) does not seem to agree with the scaling of (3.38) for $h = 0$. There are several possibilities for the disagreement¹⁸. First notice that s^3 term is analytic in $a - 1$, so this term might not be universal and does not have a Liouville interpretation. The second is that the scaling (3.38) does not hold for $\mu_0 < 0$ due to certain subtleties.

It would be very interesting to have a more complete and precise string theory identification of (3.22) and (3.23).

¹⁸ It was also pointed out to us by I. Klebanov that (3.28) has the same scaling as the $c = -2$ matrix model.

4. Discussion and conclusions

In this paper we investigated non-perturbative aspects of the Hagedorn transition for IIB string theory in an anti-de Sitter spacetime in the limit that the string length goes to infinity. We find that as one approaches the Hagedorn temperature perturbative string theory breaks down and a noncompact Liouville direction appears to open up¹⁹. In the double scaling limit $T - T_H \sim O(N^{-\frac{4}{3}})$, $N \rightarrow \infty$, the full partition function can be written as a superposition of those from the thermal AdS and the Liouville background, weighed by their relative classical action. The Hagedorn singularity of perturbative string theory around thermal AdS is cancelled by a similar divergence in the Liouville theory²⁰. Our discussions here apply to a variety of Yang-Mills theories or their string theory dual which have the same critical behavior as the double trace unitary matrix model (2.10).

The summing-over-geometry behavior we see here is rather similar to that of the Hawking-Page transition [18,19] in the strong coupling. There are some differences. One difference is that we are summing over conformal field theories instead of classical gravity backgrounds. Another important difference is that in our case at a given temperature only one background is stable. In the strong coupling limit, the Hawking-Page transition appears at a temperature $T_c \sim O(1)$, much lower than the Hagedorn temperature at $T_H \sim O(\lambda^{\frac{1}{4}})$. When extrapolated to the strong coupling, the phase transition here may become the Hawking-Page transition or describe the transition between a meta-stable thermal AdS to an AdS black hole at T_H .

It is somewhat suprising and interesting that near the transition point, the bulk theory can be described by a Liouville theory. This is reminiscent of the appearance of long throats in other singular CFTs like the conifold [37,38,39].

We also found some interesting non-renormalization properties. At temperatures below the Hagedorn temperature, the string partition function around the thermal AdS background appears to receive only one-loop contribution. Equation (3.28) also does not receive corrections beyond one loop. It would be interesting to understand these properties better.

Given the universal presence of the Hagedorn behavior in perturbative string theories, the lessons we learned here might serve as a useful guide for probing non-perturbative

¹⁹ In this section we will assume the Liouville interpretation.

²⁰ This is also reminiscent of the discussion in [36] where unitarity of the AdS black hole background is restored by summing over other geometries.

stringy effects in other string theories including those in asymptotically flat spacetime. For example, it is interesting to check whether scaling behavior exists in type II or Heterotic theory in flat spacetime as one approaches the Hagedorn temperature, i.e. we would like to know whether leading order Hagedorn divergences at higher genera have the form

$$\log Z \sim -\log(\beta - \beta_H) + \sum_{n=1}^{\infty} g_s^{2n} \frac{1}{(\beta - \beta_H)^{\gamma n}}, \quad (4.1)$$

with γ the critical exponent. We hope to come back to this question later.

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Appendix A. Mixed unitary matrix model

Here we consider the double scaling limit of the following unitary matrix model

$$Z(a, \mu) = \int dU \exp \left[a \text{Tr} U \text{Tr} U^\dagger + \frac{1}{2} N (\mu \text{Tr} U + \bar{\mu} \text{Tr} U^\dagger) \right]$$

which arises if one introduces fundamental matter [7,9]. Our discussion follows that of [24]. Introducing the Lagrange multiplier we find that

$$\begin{aligned} Z(a, \mu) &= \frac{N^2}{8\pi a} \int dU d\lambda d\bar{\lambda} \exp \left[-\frac{N^2}{4a} (\lambda - \mu)(\bar{\lambda} - \bar{\mu}) + \frac{1}{2} N (\lambda \text{Tr} U + \bar{\lambda} \text{Tr} U^\dagger) \right] \\ &= \frac{N^2}{4\pi a} \int_0^\infty g dg \int_{-\pi}^\pi d\theta \exp \left(-\frac{N^2}{4a} (g^2 - g(\mu e^{-i\theta} + \bar{\mu} e^{i\theta}) + \mu^2) + N^2 F(g) \right) \\ &= \frac{N^2}{2a} \int_0^\infty g dg e^{N^2 Q} \end{aligned} \quad (A.1)$$

where

$$N^2 Q = \log I_0 \left(\frac{N^2 g |\mu|}{2a} \right) - \frac{N^2}{4a} (|\mu|^2 + g^2) + N^2 F(g)$$

and we have used

$$e^{N^2 F(g)} = \int dU \exp \left(\frac{1}{2} N g (\text{Tr} U + \text{Tr} U^\dagger) \right)$$

and

$$\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{z \cos \theta} = I_0(z)$$

Note that we have introduced $g = |\lambda|$ and in the second line of (A.1) we absorbed the phase of μ into the integration of θ . Below for notational simplicity we will denote $|\mu|$ simply as μ . Thus $\mu > 0$.

Note that

$$\log I_0 \left(\frac{N^2 g \mu}{2a} \right) = \frac{N^2 g \mu}{2a} - \frac{1}{2} \log \left(\frac{\pi N^2 g \mu}{a} \right) - \frac{a}{4N^2 g \mu} + O(N^{-4})$$

We will evaluate (A.1) using the saddle point approximation. We will look at the leading order terms in Q . Using (3.3) we find that

1. When $a < 1$ and $\mu < \mu_0 = 1 - a$, Q has a maximum at

$$g_0 = \frac{\mu}{\mu_0} < 1, \quad Q(g_0) = \frac{\mu^2}{4(1-a)} = \frac{\mu^2}{\mu_0^2} \frac{1-a}{4}$$

2. When $a < 1$ and $\mu > \mu_0$ or $a > 1$, Q has a maximum at

$$g_0 = a + \frac{\mu}{2} + \sqrt{\left(a + \frac{1}{2}\mu\right)^2 - a}, \quad Q(g_0) = \frac{1}{2} \left(g_0 + \frac{\mu}{2a} g_0 - 1 - \log g_0 \right) - \frac{\mu^2}{4a}$$

Thus it follows from the above that in the $a - \mu$ plane, below the line $a + \mu = 1$ (note $\mu > 0$), the system is in the phase in which the distribution of eigenvalues of U has no gap on the unit circle, while above the line $a + \mu = 1$, the system develops a gap in the distribution of eigenvalues of U .

The behavior of the system as the critical line is approached can be obtained as follows. It is clear that as far as $\mu \neq 0$, the saddle points of both phases approach $g_0 = 1$ as the critical line is approached²¹. More explicitly, from the above equations one can check that from both sides

$$g_0 = 1 + \frac{\Delta}{1-a} + O(\Delta^2), \quad \Delta = \mu - \mu_0 \ll 1$$

Motivated from the scaling behavior of the unitary matrix model (3.6), we let

$$\mu = \mu_0(1 - N^{-\frac{2}{3}}s), \quad g = 1 - N^{-\frac{2}{3}}t$$

and

$$N^2 Q = K_0 - \frac{N^{\frac{2}{3}}}{4} \frac{1-a}{a} (t-s)^2 + F_0^{(2)}(t) + O(N^{-\frac{2}{3}})$$

²¹ This is different from $\mu = 0$ case, in which the saddle point is always at $g = 0$ for $a < 1$.

with

$$K_0 = \frac{N^2}{4}(1-a) + \frac{N^{\frac{4}{3}}s(1-a)}{2} + \frac{N^{\frac{2}{3}}}{4}s^2(1-a) - \frac{1}{2}\log\left(\frac{N^2\pi(1-a)}{a}\right)$$

Plugging the above expressions into (A.1) we find that

$$Z = \frac{N^{\frac{4}{3}}}{2a}e^{K_0} \int_{-\infty}^{\infty} dt e^{-\frac{N^{\frac{2}{3}}}{4}\frac{1-a}{a}(t-s)^2 + F_0^{(2)}(t)} \left(1 + O(N^{-\frac{2}{3}})\right)$$

We thus find that

$$\log Z = \tilde{K}_0 + F_0^{(2)}(s) + O(N^{-\frac{2}{3}})$$

with \tilde{K}_0 non-universal terms

$$\tilde{K}_0 = \frac{N^2}{4}(1-a) + \frac{N^{\frac{4}{3}}s(1-a)}{2} + \frac{N^{\frac{2}{3}}}{4}s^2(1-a) - \log(1-a)$$

Thus we find that with any nonzero μ , the double trace deformation of the unitary matrix model has the same critical behavior as that of the unitary matrix model up to non-universal terms. The case for $\mu = 0$ needs separate treatment, which is discussed in the main text.

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